Proof that the Derivative of an Odd Function is Even

First, we need a technical lemma (a preliminary result).

Lemma. If $\lim_{h\to 0} g(h)$ exists, then $\lim_{h\to 0} g(-h)$ exists and is equal to $\lim_{h\to 0} g(h)$.
Proof. Since

$$\lim_{h\to 0} g(h) = \lim_{h\to 0^-} g(h) = \lim_{h\to 0^+} g(-h),$$

and

$$\lim_{h\to 0} g(h) = \lim_{h\to 0^+} g(h) = \lim_{h\to 0^-} g(-h),$$

it follows that $\lim_{h\to 0} g(-h)$ exists and that

$$\lim_{h\to 0} g(-h) = \lim_{h\to 0^-} g(-h) = \lim_{h\to 0^+} g(-h) = \lim_{h\to 0} g(h). \ \square$$

Theorem. Suppose f is an odd function. Then f' (where defined) is an even function.
Proof.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-f(x+h)+f(x)}{-h} = \lim_{h\to 0} \frac{f(-(x+h))-f(-x)}{-h}$$
$$= \lim_{h\to 0} \frac{f(-x-h)-f(-x)}{-h} = \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = f'(-x),$$

where the second to last equality follows from the lemma above. $\square$